

Package ‘circularKDE’

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Title Recent Methods for Kernel Density Estimation of Circular Data

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Description Provides recent kernel density estimation methods for circular data, including adaptive and higher-order techniques. The implementation is based on recent advances in bandwidth selection and circular smoothing. Key methods include adaptive bandwidth selection methods by Zámečník et al. (2024) <[doi:10.1007/s00180-023-01401-0](https://doi.org/10.1007/s00180-023-01401-0)>, complete cross-validation by Hasilová et al. (2024) <[doi:10.59170/stattrans-2024-024](https://doi.org/10.59170/stattrans-2024-024)>, Fourier-based plug-in rules by Tenreiro (2022) <[doi:10.1080/10485252.2022.2057974](https://doi.org/10.1080/10485252.2022.2057974)>, and higher-order kernels by Tsuruta & Sagae (2017) <[doi:10.1016/j.spl.2017.08.003](https://doi.org/10.1016/j.spl.2017.08.003)>.

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URL <https://github.com/stazam/circularKDE>

BugReports <https://github.com/stazam/circularKDE/issues>

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circularKDE-package	<i>circularKDE: Recent Methods for Kernel Density Estimation of Circular Data</i>
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Description

Provides recent kernel density estimation methods for circular data, including adaptive and higher-order techniques. The implementation is based on recent advances in bandwidth selection and circular smoothing. Key methods include adaptive bandwidth selection methods by Zámečník et al. (2024) [doi:10.1007/s00180023014010](https://doi.org/10.1007/s00180023014010), complete cross-validation by Hasilová et al. (2024) [doi:10.59170/stattrans2024024](https://doi.org/10.59170/stattrans2024024), Fourier-based plug-in rules by Tenreiro (2022) [doi:10.1080/10485252.2022.2057974](https://doi.org/10.1080/10485252.2022.2057974), and higher-order kernels by Tsuruta & Sagae (2017) [doi:10.1016/j.spl.2017.08.003](https://doi.org/10.1016/j.spl.2017.08.003).

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See Also

Useful links:

- <https://github.com/stazam/circularKDE>
- Report bugs at <https://github.com/stazam/circularKDE/issues>

 adaptiveDensityCircular

Adaptive Kernel Density Estimation for circular data

Description

This function computes an adaptive kernel density estimate for circular data, adjusting the bandwidth locally based on a global bandwidth parameter and data density (see [doi:10.1007/s00180023-014010](https://doi.org/10.1007/s00180023-014010)).

Usage

```
adaptiveDensityCircular(
  x,
  bw0,
  alpha = 0.5,
  type = "n",
  z = NULL,
  from = 0,
  to = 2 * pi,
  n = 500
)
```

Arguments

x	Data for which the density is to be estimated. The object is coerced to a numeric vector in radians using <code>circular</code> . Can be a numeric vector or an object of class <code>circular</code> .
bw0	Global bandwidth parameter, a positive numeric value that sets the baseline smoothness of the density estimate. Controls the overall scale of the kernel.
alpha	Numeric scalar between 0 and 1 (default is 0.5) that determines the sensitivity of the local bandwidth adaptation. Higher values increase the influence of local data density on the bandwidth.
type	Character string specifying the method for calculating the local adaptation factor (default is "n"). Available options are: "am" Arithmetic mean of pilot density estimates "gm" Geometric mean of pilot density estimates (requires all positive values) "rv" Range (max - min) of pilot density estimates "n" No scaling factor (equivalent to fixed bandwidth)
z	Optional numeric vector of points at which to evaluate the density. If NULL (default), a grid of n equally spaced points between from and to is generated automatically.
from	Starting point of the evaluation range, a numeric value in radians. Must be finite. Default is <code>circular(0)</code> (0 radians).

to	Ending point of the evaluation range, a numeric value in radians. Must be finite. Default is <code>circular(2 * pi)</code> (2*pi radians).
n	Positive integer specifying the number of evaluation points (default is 500). Ignored if z is provided. Determines the resolution of the density estimate when z is NULL.

Details

The method extends the classical idea of variable bandwidth estimators from the linear case (Breiman et al., 1977) to the circular setting by employing the von Mises kernel. Specifically, the procedure follows three steps:

1. A pilot density estimate is obtained using a fixed global bandwidth κ (`bw0`).
2. Local adaptation factors λ_i are computed at each data point Θ_i according to

$$\lambda_i = \left\{ g / \hat{f}(\Theta_i) \right\}^\alpha,$$

where g is a global measure of central tendency (typically the geometric mean) and $\alpha \in [0, 1]$ is a sensitivity parameter.

3. The adaptive density is evaluated using local bandwidths $\lambda_i \cdot \kappa$, resulting in the estimator

$$\hat{f}(\theta) = \frac{1}{2n\pi} \sum_{i=1}^n \frac{1}{I_0(\lambda_i \kappa)} \exp(\lambda_i \kappa \cos(\theta - \Theta_i)).$$

Value

A numeric vector of length equal to the length of z (or n if z is NULL), containing the estimated density values at each evaluation point.

References

Zámečník, S., Horová, I., Katina, S., & Hasilová, K. (2023). An adaptive method for bandwidth selection in circular kernel density estimation. *Computational Statistics*. doi:10.1007/s00180023-014010

Breiman, L., Meisel, W., & Purcell, E. (1977). Variable kernel estimates of multivariate densities. *Technometrics*, 19(2), 135-144. doi:10.2307/1268623

Examples

```
# Example with numeric data in radians
library(circular)
x <- rvonmises(100, mu = circular(0), kappa = 1)
bw0 <- bwLscvg(x = x)$minimum
dens <- adaptiveDensityCircular(x, bw0 = bw0)
plot(seq(0, 2 * pi, length.out = 500), dens, type = "l",
     main = "Adaptive Circular Density")

# Example with numerical integration over interval [0,2pi] to verify normalization of
# computed density
```

```
library(circular)
x <- rvonmises(100, mu = circular(0), kappa = 1)
bw0 <- bwLscvg(x = x)$minimum
dens <- function(z)adaptiveDensityCircular(z = z, bw0 = bw0, x=x)
integrate(dens, lower = 0, upper = 2*pi) # 1 with absolute error < 4e-07
```

bwCcv

*Complete Cross-Validation for Circular Data***Description**

This function calculates the optimal smoothing parameter (bandwidth) for circular data using the complete cross-validation (CCV) method (see [doi:10.59170/stattrans2024024](https://doi.org/10.59170/stattrans2024024)).

Usage

```
bwCcv(x, lower = 0, upper = 60, tol = 0.1)
```

Arguments

x	Data from which the smoothing parameter is to be computed. The object is coerced to a numeric vector in radians using <code>circular</code> . Can be a numeric vector or an object of class <code>circular</code> .
lower	Lower boundary of the interval to be used in the search for the smoothing parameter kappa. Must be a positive numeric value less than upper. Default is 0.
upper	Upper boundary of the interval to be used in the search for the smoothing parameter kappa. Must be a positive numeric value greater than lower. Default is 60.
tol	Convergence tolerance for the <code>optimize</code> function, determining the precision of the optimization process. Also used to detect convergence near boundaries: if the optimal value is within <code>tol</code> of lower or upper, a warning is issued suggesting interval adjustment. Default is 0.1.

Details

The complete cross-validation (CCV) method is an alternative for bandwidth selection, originally proposed by Jones (1991) for linear densities. Its adaptation to the circular setting was recently studied by Hasilová et al. (2024).

The method uses functionals T_m defined as:

$$T_m(\kappa) = \frac{(-1)^m}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_\kappa^{(2m)}(\theta_i - \theta_j)$$

where $K_\kappa^{(2m)}$ is the (2m)-th derivative of K_κ .

The CCV criterion can be expressed as:

$$CCV(\kappa) = R(f(\kappa)) - T_0(\kappa) + \frac{1}{2}\bar{\sigma}_h^2 T_1(\kappa) + \frac{1}{24}(\eta_2^4(K_\kappa) - \eta_4(K_\kappa))T_2(\kappa)$$

where $\eta_j(K_\kappa)$ denotes the j -th moment of the kernel.

For the von Mises kernel, the explicit CCV criterion becomes:

$$CCV(\kappa) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n (K_\kappa * K_\kappa)(\theta_i - \theta_j) - T_0(\kappa) + \frac{A_1(\kappa)}{2\kappa} T_1(\kappa) + \frac{2A_1^2(\kappa) - A_2(\kappa)}{8\kappa^2} T_2(\kappa)$$

where $A_k(\kappa) = I_k(\kappa)/I_0(\kappa)$ is the ratio of modified Bessel functions.

The optimal bandwidth is obtained by minimizing this criterion over the interval [lower , upper].

Value

The computed optimal smoothing parameter kappa, a numeric concentration parameter (analogous to inverse radians) that minimizes the smoothed cross-validation criterion within the interval [lower , upper] and the value of objective function at that point. Higher values indicate sharper, more concentrated kernels and less smoothing; lower values indicate broader kernels and more smoothing. If the optimization fails, a warning is issued.

References

Hasilová, K., Horová, I., Valis, D., & Zámečník, S. (2024). A comprehensive exploration of complete cross-validation for circular data. *Statistics in Transition New Series*, 25(3):1–12. doi:10.59170/stattrans2024024

See Also

[bwScv](#), [bwLscvg](#), [bwTs](#)

Examples

```
# Example with circular data
library(circular)
set.seed(123)
x <- rwrappednormal(100, mu = circular(2), rho = 0.5)
bw <- bwCcv(x)
print(round(bw$minimum, 2))

x <- rvonmises(100, mu = circular(0), kappa = 1)
bw <- bwCcv(x)
print(round(bw$minimum, 2))
```

Description

This function computes the optimal smoothing parameter (bandwidth) for circular data using the Fourier series-based direct plug-in approach based on delta sequence estimators (see [doi:10.1080/10485252.2022.2057974](https://doi.org/10.1080/10485252.2022.2057974)).

Usage

```
bwFo(x, C1 = 0.25, C2 = 25, gamma = 0.5)
```

Arguments

x	Data from which the smoothing parameter is to be computed. The object is coerced to a numeric vector in radians using <code>circular</code> . Can be a numeric vector or an object of class <code>circular</code> .
C1	Numeric scalar (default 0.25) representing the lower bound constant for determining the range of Fourier coefficients. Used to compute the lower bound $L_n = \lfloor C_1 \cdot n^{1/11} \rfloor + 1$ for the optimal number of Fourier terms. Must be positive and less than C2.
C2	Numeric scalar (default 25) representing the upper bound constant for determining the range of Fourier coefficients. Used to compute the upper bound $U_n = \lfloor C_2 \cdot n^{1/11} \rfloor$ for the optimal number of Fourier terms. Must be positive and greater than C1.
gamma	Numeric scalar between 0 and 1 (default 0.5) representing the penalty parameter in the criterion function $H(m)$ used for selecting the optimal number of Fourier coefficients. Controls the trade-off between bias and variance in the functional estimation.

Details

The Fourier-based plug-in estimator computes the optimal bandwidth using the formula:

$$\hat{h}_{FO} := (4\pi)^{-1/10} \hat{\theta}_{2, \hat{m}}^{-1/5} n^{-1/5}$$

where $\hat{\theta}_{2, \hat{m}}$ is the estimator of the second-order functional $\theta_2(f)$ based on the selected number of Fourier coefficients \hat{m} .

Under the assumption of von Mises density, this formula becomes:

$$\hat{h}_{VM} = (4\pi)^{-1/10} \left(\frac{3\hat{\kappa}^2 I_0(2\hat{\kappa}) - \hat{\kappa} I_1(2\hat{\kappa})}{8\pi I_0(\hat{\kappa})^2} \right)^{-1/5} n^{-1/5}$$

where I_0 and I_1 are the modified Bessel functions of the first kind of orders 0 and 1, and $\hat{\kappa}$ is the estimated concentration parameter of the von Mises distribution.

Value

The computed optimal smoothing parameter kappa, a numeric concentration parameter (analogous to inverse radians) derived from the Fourier method for circular kernel density estimation. Higher values indicate sharper, more concentrated kernels and less smoothing; lower values indicate broader kernels and more smoothing.

References

Tenreiro, C. (2022). Kernel density estimation for circular data: a Fourier series-based plug-in approach for bandwidth selection. *Journal of Nonparametric Statistics*, 34(2):377–406. doi:[10.1080/10485252.2022.2057974](https://doi.org/10.1080/10485252.2022.2057974)

See Also

[bwScv](#), [bwLscvg](#), [bwCcv](#)

Examples

```
# Example with circular data
library(circular)
set.seed(123)
x <- rvonmises(100, mu = circular(0), kappa = 2)
bw <- bwFo(x)
print(bw)

x <- rwrappednormal(100, mu = circular(1), rho = 0.7)
bw <- bwFo(x)
y <- density(x, bw=bw)
plot(y, main="KDE with Fourier Plug-in Bandwidth")
```

 bwJf

Plug-in Method by Tsuruta and Sagae with additive Jones-Foster approach

Description

This function computes the optimal smoothing parameter (bandwidth) for circular data using the plug-in method introduced by Tsuruta and Sagae (see [doi:10.1016/j.spl.2017.08.003](https://doi.org/10.1016/j.spl.2017.08.003)) with the additive method from Jones and Foster (1993) to form higher-order kernel functions.

Usage

```
bwJf(x, verbose = FALSE)
```

Arguments

x	Data from which the smoothing parameter is to be computed. The object is coerced to a numeric vector in radians using <code>circular</code> . Can be a numeric vector or an object of class <code>circular</code> .
verbose	Logical indicating whether to print intermediate computational values for debugging and teaching purposes. Shows <code>kappa_hat</code> , <code>r_hat</code> , and component calculations. Default is <code>FALSE</code> .

Details

The plug-in approach estimates the optimal bandwidth through the following steps:

1. Apply the additive method from Jones and Foster (1993) to construct a p-th order kernel function.
2. Derive expression for asymptotic mean integrated squared error (AMISE) expression.
3. Solving for the bandwidth that minimizes the AMISE. The optimal bandwidth for the additive Jones-Foster method is given by:

$$\hat{\kappa}_{JF} = \left[\frac{16\sqrt{\pi}}{3} \hat{R}_{\hat{\tau}} \left(\frac{5f_{VM}^{(2)} + 2f_{VM}^{(4)}}{12} \right) n \right]^{2/9}$$

where the functional $\hat{R}_{\hat{\tau}}$ is computed as a weighted linear combination under the von Mises assumption and $\hat{\tau}$ is the MLE estimate of the von Mises concentration parameter used as the initial value.

Value

The computed optimal smoothing parameter `kappa`, a numeric concentration parameter (analogous to inverse radians) derived from the circular version of the additive method for circular kernel density estimation. Higher values indicate sharper, more concentrated kernels and less smoothing; lower values indicate broader kernels and more smoothing.

References

- Tsuruta, Yasuhito & Sagae, Masahiko (2017). Higher order kernel density estimation on the circle. *Statistics & Probability Letters*, 131:46–50. doi:10.1016/j.spl.2017.08.003
- Jones, M. C. & Foster, P. J. (1993). Generalized jackknifing and higher-order kernels. *Journal of Nonparametric Statistics*, 3:81–94. doi:10.1080/10485259308832573

See Also

[bwScv](#), [bwLscvg](#), [bwCcv](#)

Examples

```
# Example with circular data
library(circular)
set.seed(123)
x <- rvonmises(100, mu = circular(0), kappa = 2)
bw <- bwJf(x)
print(bw)

x <- rwrappednormal(100, mu = circular(1), rho = 0.7)
bw <- bwJf(x)
print(bw)
```

bwLscvg

Generalized Least Squares Cross-Validation for Circular Data

Description

This function computes the optimal smoothing parameter (bandwidth) for circular data using a generalized least squares cross-validation method (see [doi:10.1007/s00180023014010](https://doi.org/10.1007/s00180023014010)).

Usage

```
bwLscvg(x, g = NULL, lower = 0, upper = 60, tol = 0.1)
```

Arguments

- | | |
|-------|--|
| x | Data from which the smoothing parameter is to be computed. The object is coerced to a numeric vector in radians using <code>circular</code> . Can be a numeric vector or an object of class <code>circular</code> . |
| g | A numeric scalar that controls the variability in the cross-validation procedure. It influences the scaling in the internal calculations, affecting the bandwidth estimation. It needs to be positive number not equal to 2. Default is 4. |
| lower | Lower boundary of the interval to be used in the search for the smoothing parameter κ . Must be a positive numeric value less than upper. Default is 0. |
| upper | Upper boundary of the interval to be used in the search for the smoothing parameter κ . Must be a positive numeric value greater than lower. Default is 60. |
| tol | Convergence tolerance for the <code>optimize</code> function, determining the precision of the optimization process. Default is 0.1. |

Details

The generalized least squares cross-validation method (LSCV_g) is an adaptation of the method originally introduced by Zhang for linear data, developed for circular data (see Zamecnik, et.al 2025) to address the finite-sample performance issues of standard LSCV.

The LSCV_g criterion is defined as:

$$LSCV_g(\kappa) = \frac{1}{n}R(K_\kappa) + \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \left(\frac{n-1}{n} (K_\kappa * K_\kappa)(\theta_i - \theta_j) + \frac{2}{g(g-2)} K_{\kappa/g}(\theta_i - \theta_j) - \frac{g-1}{g-2} K_{\kappa/2}(\theta_i - \theta_j) \right)$$

Using the von Mises kernel, this takes the computational form:

$$LSCV_g(\kappa) = \frac{1}{2\pi n^2} \sum_{i=1}^n \sum_{j=1}^n \frac{I_0(\kappa \sqrt{2(1 + \cos(\theta_i - \theta_j))})}{I_0^2(\kappa)} + \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \left(\frac{2}{g(g-2)} \frac{\exp(\frac{\kappa}{g} \cos(\theta_i - \theta_j))}{2\pi I_0(\kappa/g)} - \frac{g-1}{g-2} \frac{\exp(\frac{\kappa}{2} \cos(\theta_i - \theta_j))}{2\pi I_0(\kappa/2)} \right)$$

The optimal bandwidth is obtained by minimizing this criterion over the interval [lower , upper].

Value

The computed optimal smoothing parameter kappa, a numeric concentration parameter (analogous to inverse radians) that minimizes the smoothed cross-validation criterion within the interval [lower , upper] and the value of objective function at that point. Higher values indicate sharper, more concentrated kernels and less smoothing; lower values indicate broader kernels and more smoothing. If the optimization fails, a warning is issued.

References

- Zámečník, S., Horová, I., & Hasilová, K. (2025). Generalised least square cross-validation for circular data. *Communications in Statistics*. doi:10.1007/s00180023014010
- Zhang, J. (2015). Generalized least squares cross-validation in kernel density estimation. *Statistica Neerlandica*, 69(3), 315-328. doi:10.1111/stan.12061

See Also

[bwScv](#), [bwFo](#), [bwCcv](#)

Examples

```
# Example with circular data
library(circular)
set.seed(123)
x <- rwrappednormal(100, mu = circular(2), rho = 0.5)
bw <- bwLscvg(x)
print(round(bw$minimum, 2))

x <- rvonmises(100, mu = circular(0), kappa = 1)
bw <- bwLscvg(x)
print(round(bw$minimum, 2))
plot(density.circular(x, bw = bw$minimum))
```

bwScv

*Smoothed Cross-Validation for Circular Data***Description**

This function computes the optimal smoothing parameter (bandwidth) for circular data using a smoothed cross-validation (SCV) method (see [doi:10.1007/s00180-023-01401-0](https://doi.org/10.1007/s00180-023-01401-0)).

Usage

```
bwScv(x, np = 500, lower = 0, upper = 60, tol = 0.1)
```

Arguments

x	Data from which the smoothing parameter is to be computed. The object is coerced to a numeric vector in radians using <code>circular</code> . Can be a numeric vector or an object of class <code>circular</code> . Note: computational complexity scales as $O(n^2)$ due to outer product operations; datasets with $n > 2000$ may cause performance and memory issues.
np	An integer specifying the number of points used in numerical integration to evaluate the SCV criterion. A higher number increases precision but also computational cost (recommended value is ≥ 100). Default is 500.
lower	Lower boundary of the interval for the optimization of the smoothing parameter κ . Must be a positive numeric value smaller than upper. Default is 0.
upper	Upper boundary of the interval for the optimization of the smoothing parameter κ . Must be a positive numeric value greater than lower. Default is 60.
tol	Convergence tolerance for the <code>optimize</code> function, determining the precision of the optimization process. Also used to detect convergence near boundaries: if the optimal value is within <code>tol</code> of lower or upper, a warning is issued suggesting interval adjustment. Default is 0.1.

Details

The smoothed cross-validation (SCV) method is an alternative bandwidth selection approach, originally introduced by Hall & Marron (1992) for linear densities and adapted for circular data by Zámečník et al. (2023).

The SCV criterion is given by

$$\text{SCV}(\kappa) = \frac{R(K)}{nh} + \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n (K_{\kappa} * K_{\kappa} * K_{\kappa} * K_{\kappa} - 2K_{\kappa} * K_{\kappa} * K_{\kappa} + K_{\kappa} * K_{\kappa})(\Theta_i - \Theta_j)$$

where K_{κ} is the Von Mises kernel with concentration κ (for the formula see 3.7, 3.8 in Zámečník et al. (2023)). The optimal bandwidth minimizes the sum $ISB(\kappa) + IV(\kappa)$ over the interval `[lower, upper]`.

The integral expressions involved in the SCV criterion (see Sections 3.2 in Zámečník et al., 2023) are evaluated numerically using the trapezoidal rule on a uniform grid of length `np`.

Value

The computed optimal smoothing parameter kappa, a numeric concentration parameter (analogous to inverse radians) that minimizes the smoothed cross-validation criterion within the interval [lower, upper] and the value of objective function at that point. Higher values indicate sharper, more concentrated kernels and less smoothing; lower values indicate broader kernels and more smoothing. If the optimization fails, a warning is issued.

References

- Zámečník, S., Horová, I., Katina, S., & Hasilová, K. (2023). An adaptive method for bandwidth selection in circular kernel density estimation. *Computational Statistics*. doi:10.1007/s00180023-014010
- Hall, P., & Marron, J. S. (1992). On the amount of noise inherent in bandwidth selection for a kernel density estimator. *The Annals of Statistics*, 20(1), 163-181.

See Also

[bwTs](#), [bwLscvg](#), [bwCcv](#)

Examples

```
# Example with circular data (Lower `nu` = more smoothing; higher = sharper peaks).
library(circular)
x <- rwrappednormal(100, mu = circular(2), rho = 0.5)
bw <- bwScv(x)
print(round(bw$minimum, 2))

x <- rvonmises(100, mu = circular(0.5), kappa = 2)
bw <- bwScv(x)
print(round(bw$minimum, 2))
```

 bwTs

Plug-in Method by Tsuruta and Sagae with multiplicative method from Terrell and Scott

Description

This function computes the optimal smoothing parameter (bandwidth) for circular data using the plug-in method, introduced by Tsuruta and Sagae (see doi:10.1016/j.spl.2017.08.003) with the multiplicative method from Terrell and Scott (1980) to form higher-order kernel functions. The method optimally balances the bias-variance tradeoff by minimizing asymptotic mean integrated squared error.

Usage

```
bwTs(x, verbose = FALSE)
```

Arguments

x	Data from which the smoothing parameter is to be computed. The object is coerced to a numeric vector in radians using <code>conversion.circular</code> . Can be a numeric vector or an object of class <code>circular</code> .
verbose	Logical indicating whether to print intermediate computational values for debugging purposes. Useful for diagnosing floating-point instability in Bessel-based moment terms. Default is FALSE.

Details

The plug-in approach estimates the optimal bandwidth through the following steps:

1. Apply the multiplicative method from Terrell and Scott (1980) to construct a p-th order kernel function.
2. Derive expression for asymptotic mean integrated squared error (AMISE) expression.
3. Solving for the bandwidth that minimizes the AMISE. The optimal bandwidth for the multiplicative Terrell-Scott method is given by:

$$\hat{h}_{TS} = \left[\frac{288}{33 - 16\sqrt{2/5}} \hat{R}_{\hat{\tau}}(m_{VM})n \right]^{2/9}$$

where the computational formula is:

$$m_{VM}(\theta) := [2\{f_{VM}^{(2)}\}^2 / f_{VM} - 5f_{VM}^{(2)} + 2f_{VM}^{(4)}] / 4$$

and $\hat{R}_{\hat{\tau}}(m_{VM})$ is the functional computed under the von Mises assumption using the multiplicative approach. The parameter $\hat{\tau}$ is the MLE estimate of the von Mises concentration parameter used as the initial value.

Value

The computed optimal smoothing parameter, a numeric value derived from the circular version of the multiplicative method for circular kernel density estimation.

References

- Tsuruta, Yasuhito & Sagae, Masahiko (2017). Higher order kernel density estimation on the circle. *Statistics & Probability Letters*, 131:46–50. doi:10.1016/j.spl.2017.08.003
- Terrell, George R. & Scott, David W. (1980). On improving convergence rates for nonnegative kernel density estimators. *The Annals of Statistics*, 8(5):1160–1163.

See Also

[bwScv](#), [bwLscvg](#), [bwCcv](#)

Examples

```
# Example with circular data
library(circular)
set.seed(123)
x <- rvonmises(100, mu = circular(0), kappa = 2)
bw <- bwTs(x)
print(bw)

x <- rwrappednormal(100, mu = circular(1), rho = 0.7)
x <- as.numeric(x)
bw <- bwTs(x)
print(bw)
```

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