Package 'qfa'

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Type Package

Title Quantile-Frequency Analysis (QFA) of Time Series and Spline Quantile Regression (SQR)

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Description Implementation of quantile frequency analysis (QFA) for time series based on trigonometric quantile regression and of spline quantile regression (SQR) for estimating the coefficients in linear quantile regression models as smooth functions of the quantile level.
References:

[1] Li, T.-H. (2012). "Quantile periodograms," J. of the American Statistical Association, 107, 765–776. <doi:10.1080/01621459.2012.682815>

[2] Li, T.-H. (2014). Time Series with Mixed Spectra, CRC Press. <doi:10.1201/b15154>

[3] Li, T.-H. (2025). "Quantile Fourier transform, quantile series, and nonparametric estimation of quantile spectra," Communications in Statistics: Simulation and Computation, 1–22. <doi:10.1080/03610918.2025.2509820>

[4] Li, T.-H. (2025). "Quantile-crossing spectrum and spline autoregression estimation," Statistical Inference for Stochastic Processes, 28, 20. <doi:10.1007/s11203-025-09336-7>

[5] Li, T.-H. (2025). "Spline autoregression method for estimation of quantile spectrum," J. of Computational and Graphical Statistics, 1-15. <doi:10.1080/10618600.2025.2549452>

[6] Li, T.-H., and Megiddo, N. (2026). "Spline quantile regression," J. of Statistical Theory and Practice, 20, 30. <doi:10.1007/s42519-026-00545-8>

[7] Li, T.-H. (2026). "Spline quantile regression with cubic and linear smoothing splines," <doi:10.48550/arXiv.2603.22408>.

Depends R (>= 3.5)

Imports RhpcBLASctl, doParallel, fields, foreach, stats, Matrix, SparseM, methods, mgcv, nlme, parallel, quantreg, splines, graphics, colorRamps, MASS, osqp, piqp, boot

License GPL (>= 2)

URL https://github.com/IBM/qfa, https://github.com/thl2019/QFA

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Repository CRAN

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Contents

`birthweight` *Birth data*

Description

A random sample of US birth data in 2022 (birth weight and covariates). `Precare` and `Education` should be treated as factors.

Usage

```
data(birthweight)
```

Format

An object of class `data.frame` with 50000 rows and 12 columns.

Source

Data file https://data.nber.org/nvss/natality/csv/2022/natality2022us.csv from NBER https://www.nber.org/research/data/vital-statistics-natality-birth-data. Original source from the National Center for Health Statistics https://www.cdc.gov/nchs/nvss/births.htm.

References

Koenker, R. (2005). Quantile Regression. Cambridge University Press.

`boot.sqr` *Bootstrap of Spline Quantile Regression (SQR) Coefficients*

Description

This function generates bootstrap samples for the SQR coefficients and their derivatives

Usage

```
boot.sqr(
  fit,
  nsim = 1000,
  blocklength = 1,
  n.cores = 1,
  mthreads = FALSE,
  seed = 1234567
)
```

Arguments

fit	output from sqr()
nsim	number of bootstrap samples (default = 1000)
blocklength	blocklength for block sampling scheme (default = 1)
n.cores	number of cores for parallel computing (default = 1)
mthreads	if FALSE (default), set RhpcBLASctl::blas_set_num_threads(1)
seed	seed of random number generator (default = 1234567)

Value

A list with the following elements:

coefficients	array of bootstrap regression coefficients
derivatives	array of boostrap derivatives of regression coefficient
info	sequence convergence status

---

engel *Engel's food expenditure data*

Description

Engel's food expenditure data from the R package 'quantreg'.

Usage

```
data(engel)
```

Format

An object of class data.frame with 235 rows and 2 columns.

References

Koenker, R. (2005). Quantile Regression. Cambridge University Press.

finIndex *Financial index*

Description

Daily closing values of DJIA and FTSE 100 on trading days from 2001-11-27 to 2010-03-10.

Usage

```
data(finIndex)
```

Format

An object of class `data.frame` with 2048 rows and 3 columns.

Source

https://www.wsj.com/market-data/quotes/index/DJIA/historical-prices and https://www.wsj.com/market-data/quotes/index/UK/UKX/historical-prices

per *Periodogram (PER)*

Description

This function computes the periodogram or periodogram matrix for univariate or multivariate time series.

Usage

```
per(y)
```

Arguments

y	vector or matrix of time series s (if matrix, nrow(y) = length of time series)

Value

vector or array of periodogram

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y.per <- per(y)
plot(y.per)
```

qacf *Quantile Autocovariance Function (QACF)*

Description

This function computes quantile autocovariance function (QACF) from time series or quantile discrete Fourier transform (QDFT).

Usage

```
qacf(y, tau, y.qdft = NULL, n.cores = 1, cl = NULL)
```

Arguments

y	vector or matrix of time series (if matrix, nrow(y) = length of time series)
tau	sequence of quantile levels in (0,1)
y.qdft	matrix or array of pre-calculated QDFT (default = NULL: compute from y and tau); if y.qdft is supplied, y and tau can be left unspecified
n.cores	number of cores for parallel computing of QDFT if y.qdft = NULL (default = 1)
cl	pre-existing cluster for repeated parallel computing of QDFT (default = NULL)

Value

matrix or array of quantile autocovariance function

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
# compute from time series
y.qacf <- qacf(y,tau)
# compute from QDFT
y.qdft <- qdft(y,tau)
y.qacf <- qacf(y.qdft=y.qdft)
```

qcser *Quantile-Crossing Series (QCSER)*

Description

This function creates the quantile-crossing series (QCSER) for univariate or multivariate time series.

Usage

```
qcser(y, tau, normalize = FALSE)
```

Arguments

y	vector or matrix of time series
tau	sequence of quantile levels in (0,1)
normalize	TRUE or FALSE (default): normalize QCSER to have unit variance

Value

A matrix or array of quantile-crossing series

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.qser <- qcser(y,tau)
dim(y.qser)
```

---

qdft *Quantile Discrete Fourier Transform (QDFT)*

Description

This function computes quantile discrete Fourier transform (QDFT) for univariate or multivariate time series.

Usage

```
qdft(y, tau, n.cores = 1, cl = NULL)
```

Arguments

y	vector or matrix of time series (if matrix, nrow(y) = length of time series)
tau	sequence of quantile levels in (0,1)
n.cores	number of cores for parallel computing (default = 1)
cl	pre-existing cluster for repeated parallel computing (default = NULL)

Value

matrix or array of quantile discrete Fourier transform of y

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.qdft <- qdft(y,tau)
# Make a cluster for repeated use
n.cores <- 2
cl <- parallel::makeCluster(n.cores)
parallel::clusterExport(cl, c("tqr.fit"))
doParallel::registerDoParallel(cl)
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y.qdft <- qdft(y1,tau,n.cores=n.cores,cl=cl)
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y.qdft <- qdft(y2,tau,n.cores=n.cores,cl=cl)
parallel::stopCluster(cl)
```

---

qdft2qacf *Quantile Autocovariance Function (QACF)*

Description

This function computes quantile autocovariance function (QACF) from QDFT.

Usage

```
qdft2qacf(y.qdft, return.qser = FALSE)
```

Arguments

`y.qdft`	matrix or array of QDFT from `qdft()`
`return.qser`	if `TRUE`, return quantile series (QSER) along with QACF

Value

matrix or array of quantile autocovariance function if `return.sqer = FALSE` (default), else a list with the following elements:

`qacf`	matirx or array of quantile autocovariance function
`qser`	matrix or array of quantile series

Examples

```
# single time series
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.qdft <- qdft(y1,tau)
y.qacf <- qdft2qacf(y.qdft)
plot(c(0:9),y.qacf[c(1:10),1],type='h',xlab="LAG",ylab="QACF")
y.qser <- qdft2qacf(y.qdft,return.qser=TRUE)$qser
plot(y.qser[,1],type='l',xlab="TIME",ylab="QSER")
```

```
# multiple time series
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
y.qdft <- qdft(cbind(y1,y2),tau)
y.qacf <- qdft2qacf(y.qdft)
plot(c(0:9),y.qacf[1,2,c(1:10),1],type='h',xlab="LAG",ylab="QACF")
```

---

qdft2qper *Quantile Periodogram (QPER)*

Description

This function computes quantile periodogram (QPER) from QDFT.

Usage

```
qdft2qper(y.qdft)
```

Arguments

y.qdft	matrix or array of QDFT from qdft()

Value

matrix or array of quantile periodogram

Examples

```
# single time series
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.qdft <- qdft(y1,tau)
y.qper <- qdft2qper(y.qdft)
n <- length(y1)
ff <- c(0:(n-1))/n
sel.f <- which(ff > 0 & ff < 0.5)
qfa.plot(ff[sel.f],tau,Re(y.qper[sel.f,]))
# multiple time series
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
y.qdft <- qdft(cbind(y1,y2),tau)
y.qper <- qdft2qper(y.qdft)
qfa.plot(ff[sel.f],tau,Re(y.qper[1,1,sel.f,]))
qfa.plot(ff[sel.f],tau,Re(y.qper[1,2,sel.f,]))
```

qdft2qser *Quantile Series (QSER)*

Description

This function computes quantile series (QSER) from QDFT.

Usage

```
qdft2qser(y.qdft)
```

Arguments

y.qdft	matrix or array of QDFT from qdft()

Value

matrix or array of quantile series

Examples

```
# single time series
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.qdft <- qdft(y1,tau)
y.qser <- qdft2qser(y.qdft)
plot(y.qser[,1],type='l',xlab="TIME",ylab="QSER")
# multiple time series
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
y.qdft <- qdft(cbind(y1,y2),tau)
y.qser <- qdft2qser(y.qdft)
plot(y.qser[1,,1],type='l',xlab="TIME",ylab="QSER")
```

qfa.plot *Quantile-Frequency Plot*

Description

This function creates an image plot of quantile spectrum.

Usage

```
qfa.plot(
  freq,
  tau,
  rqper,
  rg.qper = range(rqper),
  rg.tau = range(tau),
  rg.freq = c(0, 0.5),
  color = colorRamps::matlab.like2(1024),
  ylab = "QUANTILE LEVEL",
  xlab = "FREQUENCY",
  tlab = NULL,
  set.par = TRUE,
  legend.plot = TRUE,
  line = 0.5
)
```

Arguments

freq	sequence of frequencies in (0,0.5) at which quantile spectrum is evaluated
tau	sequence of quantile levels in (0,1) at which quantile spectrum is evaluated
rqper	real-valued matrix of quantile spectrum evaluated on the freq x tau grid
rg.qper	zlim for qper (default = range(qper))
rg.tau	ylim for tau (default = range(tau))
rg.freq	xlim for freq (default = c(0,0.5))
color	colors (default = colorRamps::matlab.like2(1024))
ylab	label of y-axis (default = "QUANTILE LEVEL")
xlab	label of x-axis (default = "FREQUENCY")
tlab	title of plot (default = NULL)
set.par	if TRUE, par() is set internally (single image)
legend.plot	if TRUE, legend plot is added
line	distance between the image and the axis (default = 0.5)

Value

no return value

qkl.divergence *Kullback-Leibler Divergence of Quantile Spectral Estimate*

Description

This function computes Kullback-Leibler divergence (KLD) of quantile spectral estimate.

Usage

```
qkl.divergence(y.qper, qspec, sel.f = NULL, sel.tau = NULL)
```

Arguments

y.qper	matrix or array of quantile spectral estimate from, e.g., qspec.lw()
qspec	matrix of array of true quantile spectrum (same dimension as y.qper)
sel.f	index of selected frequencies for computation (default = NULL: all frequencies)
sel.tau	index of selected quantile levels for computation (default = NULL: all quantile levels)

Value

real number of Kullback-Leibler divergence

qper *Quantile Periodogram (QPER)*

Description

This function computes quantile periodogram (QPER) from time series or quantile discrete Fourier transform (QDFT).

Usage

```
qper(y, tau, y.qdft = NULL, n.cores = 1, cl = NULL)
```

Arguments

y	vector or matrix of time series (if matrix, nrow(y) = length of time series)
tau	sequence of quantile levels in (0,1)
y.qdft	matrix or array of pre-calculated QDFT (default = NULL: compute from y and tau); if y.qdft is supplied, y and tau can be left unspecified
n.cores	number of cores for parallel computing of QDFT if y.qdft = NULL (default = 1)
cl	pre-existing cluster for repeated parallel computing of QDFT (default = NULL)

Value

matrix or array of quantile periodogram

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
# compute from time series
y.qper <- qper(y,tau)
# compute from QDFT
y.qdft <- qdft(y,tau)
y.qper <- qper(y.qdft=y.qdft)
```

---

qper2 *Quantile Periodogram Type II (QPER2)*

Description

This function computes type-II quantile periodogram for univariate time series.

Usage

```
qper2(y, freq, tau, weights = NULL, n.cores = 1, cl = NULL)
```

Arguments

y	univariate time series
freq	sequence of frequencies in [0,1)
tau	sequence of quantile levels in (0,1)
weights	sequence of weights in quantile regression (default = NULL: weights equal to 1)
n.cores	number of cores for parallel computing (default = 1)
cl	pre-existing cluster for repeated parallel computing (default = NULL)

Value

matrix of quantile periodogram evaluated on freq * tau grid

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
n <- length(y)
ff <- c(0:(n-1))/n
sel.f <- which(ff > 0 & ff < 0.5)
y.qper2 <- qper2(y,ff,tau)
qfa.plot(ff[sel.f],tau,Re(y.qper2[sel.f,]))
```

qser *Quantile Series (QSER)*

Description

This function computes quantile series (QSER) from time series or quantile discrete Fourier transform (QDFT).

Usage

```
qser(y, tau, y.qdft = NULL, n.cores = 1, cl = NULL)
```

Arguments

y	vector or matrix of time series (if matrix, nrow(y) = length of time series)
tau	sequence of quantile levels in (0,1)
y.qdft	matrix or array of pre-calculated QDFT (default = NULL: compute from y and tau); if y.qdft is supplied, y and tau can be left unspecified
n.cores	number of cores for parallel computing of QDFT if y.qdft = NULL (default = 1)
cl	pre-existing cluster for repeated parallel computing of QDFT (default = NULL)

Value

matrix or array of quantile series

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
# compute from time series
y.qser <- qser(y,tau)
# compute from QDFT
y.qdft <- qdft(y,tau)
y.qser <- qser(y.qdft=y.qdft)
```

qser2ar *Autoregression (AR) Model of Quantile Series*

Description

This function fits an autoregression (AR) model to quantile series (QSER) separately for each quantile level using stats::ar().

Usage

```
qser2ar(
  y.qser,
  p = NULL,
  order.max = NULL,
  method = c("none", "gamm", "sp"),
  spar = NULL,
  type = c(1, 2)
)
```

Arguments

y.qser	matrix or array of pre-calculated QSER, e.g., using qser()
p	order of AR model (default = NULL: selected by AIC)
order.max	maximum order for AIC if p = NULL (default = NULL: determined by stats::ar())
method	quantile smoothing method: "gamm" for mgcv::gamm(), "sp" for stats::smooth.spline(), or "none" (default)
spar	smoothing parameter for stats::smooth.spline() (default = NULL for GCV)
type	type of smoothing for covariance matrix: 1 (direct, default) or 2 (square root)

Value

a list with the following elements:

A	matrix or array of AR coefficients
V	vector or matrix of residual covariance
p	order of AR model
n	length of time series
residuals	matrix or array of residuals

---

qser2qacf *ACF of Quantile Series (QSER) or Quantile-Crossing Series (QCACF)*

Description

This function creates the ACF of quantile series or quantile-crossing series

Usage

```
qser2qacf(y.qser)
```

Arguments

y.qser	matrix or array of quantile-crossing series

Value

A matrix or array of ACF

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.qser <- qcser(y,tau)
y.qacf <- qser2qacf(y.qser)
dim(y.qacf)
```

`qser2sar` *Spline Autoregression (SAR) Model of Quantile Series*

Description

This function fits spline autoregression (SAR) model to quantile series (QSER).

Usage

```
qser2sar(
  y.qser,
  tau0,
  tau = tau0,
  p = NULL,
  order.max = NULL,
  spar = NULL,
  method = c("GCV", "AIC", "BIC"),
  type = c(1, 2),
  weights = rep(1, length(tau0)),
  interval = c(-1.5, 1.5)
)
```

Arguments

`y.qser`	matrix or array of pre-calculated QSER at `tau0` using, e.g., `qser()`
`tau0`	quantile levels used to compute `y.qser`
`tau`	quantile levels for evaluation (`min(tau0) <= tau <= max(tau0)`; default = `tau0`)
`p`	order of SAR model (default = `NULL`: automatically selected by AIC)
`order.max`	maximum order for AIC if `p` = `NULL` (default = `NULL`: determined by `stats::ar()`)
`spar`	penalty parameter alla `stats::smooth.spline()` (default = `NULL`: automatically selected)
`method`	criterion for penalty parameter selection: `"GCV"` (default), `"AIC"`, or `"BIC"`

type	type of quantile smoothing for residual variance: 1 = direct (default) or 2 = square root
weights	sequence of weights (default = rep(1,length(tau0)))
interval	interval for spar optimization (default = c(-1.5,1.5))

Value

a list with the following elements:

A	matrix or array of SAR coefficients
V	vector or matrix of SAR residual covariance
p	order of SAR model
spar	penalty parameter
n	length of time series
tau	quantile levels for evaluation
tau0	quantile levels for fitting
weights	weights in penalty function
sdm	spline design matrices
fit	object containing details of SAR fit

---

qspec.ar *Autoregression (AR) Estimator of Quantile Spectrum*

Description

This function computes autoregression (AR) estimate of quantile spectrum from time series or quantile series (QSER).

Usage

```
qspec.ar(
  y,
  tau,
  y.qser = NULL,
  p = NULL,
  order.max = NULL,
  freq = NULL,
  method = c("none", "gamm", "sp"),
  spar = NULL,
  type = c(1, 2),
  n.cores = 1,
  cl = NULL
)
```

Arguments

y	vector or matrix of time series (if matrix, nrow(y) = length of time series)
tau	sequence of quantile levels in (0,1)
y.qser	matrix or array of pre-calculated QSER (default = NULL: compute from y and tau); if y.qser is supplied, y and tau can be left unspecified
p	order of AR model (default = NULL: automatically selected by AIC)
order.max	maximum order for AIC if p = NULL (default = NULL: determined by stats::ar())
freq	sequence of frequencies in [0,1) (default = NULL: all Fourier frequencies)
method	method of quantile smoothing: "gamm" for mgcv::gamm(), "sp" for stats::smooth.spline(), or "none" (default)
spar	smoothing parameter for stats::smooth.spline() (default = NULL for GCV)
type	type of smoothing for covariance matrix: 1 (direct, default) or 2 (square root)
n.cores	number of cores for parallel computing of QDFT if y.qser = NULL (default = 1)
cl	pre-existing cluster for repeated parallel computing of QDFT (default = NULL)

Value

a list with the following elements:

spec	matrix or array of AR quantile spectrum
freq	sequence of frequencies
fit	object of AR model
qser	matrix or array of quantile series if y.qser = NULL

Examples

```
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
y <- cbind(y1,y2)
tau <- seq(0.1,0.9,0.05)
n <- length(y1)
ff <- c(0:(n-1))/n
sel.f <- which(ff > 0 & ff < 0.5)
y.qspec.ar <- qspec.ar(y,tau,p=1)$spec
qfa.plot(ff[sel.f],tau,Re(y.qspec.ar[1,1,sel.f,]))
y.qser <- qcser(y1,tau)
y.qspec.ar <- qspec.ar(y.qser=y.qser,p=1)$spec
qfa.plot(ff[sel.f],tau,Re(y.qspec.ar[sel.f,]))
y.qspec.arqs <- qspec.ar(y.qser=y.qser,p=1,method="sp")$spec
qfa.plot(ff[sel.f],tau,Re(y.qspec.arqs[sel.f,]))
```

qspec.lw *Lag-Window (LW) Estimator of Quantile Spectrum*

Description

This function computes lag-window (LW) estimate of quantile spectrum with or without quantile smoothing from time series or quantile autocovariance function (QACF).

Usage

```
qspec.lw(
  y,
  tau,
  y.qacf = NULL,
  M = NULL,
  method = c("none", "gamm", "sp"),
  spar = "GCV",
  n.cores = 1,
  cl = NULL
)
```

Arguments

y	vector or matrix of time series (if matrix, nrow(y) = length of time series)
tau	sequence of quantile levels in (0,1)
y.qacf	matrix or array of pre-calculated QACF (default = NULL: compute from y and tau); if y.qacf is supplied, y and tau can be left unspecified
M	bandwidth parameter of lag window (default = NULL: quantile periodogram)
method	quantile smoothing method: "gamm" for mgcv::gamm(), "sp" for stats::smooth.spline(), or "none" (default)
spar	smoothing parameter in smooth.spline() if method = "sp" (default = "GCV")
n.cores	number of cores for parallel computing (default = 1)
cl	pre-existing cluster for repeated parallel computing (default = NULL)

Value

A list with the following elements:

spec	matrix or array of spectral estimate
spec.lw	matrix or array of spectral estimate without quantile smoothing
lw	lag-window sequence
qacf	matrix or array of quantile autocovariance function if y.qacf = NULL

Examples

```
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
tau <- seq(0.1,0.9,0.05)
n <- length(y1)
ff <- c(0:(n-1))/n
sel.f <- which(ff > 0 & ff < 0.5)
y.qacf <- qacf(cbind(y1,y2),tau)
y.qper.lw <- qspec.lw(y.qacf=y.qacf,M=5)$spec
qfa.plot(ff[sel.f],tau,Re(y.qper.lw[1,1,sel.f,]))
y.qper.lwqs <- qspec.lw(y.qacf=y.qacf,M=5,method="sp",spar=0.9)$spec
qfa.plot(ff[sel.f],tau,Re(y.qper.lwqs[1,1,sel.f,]))
```

`qspec.sar` *Spline Autoregression (SAR) Estimator of Quantile Spectrum*

Description

This function computes spline autoregression (SAR) estimate of quantile spectrum.

Usage

```
qspec.sar(
  y,
  y.qser = NULL,
  tau0,
  tau = tau0,
  p = NULL,
  order.max = NULL,
  spar = NULL,
  method = c("GCV", "AIC", "BIC"),
  type = c(1, 2),
  weights = rep(1, length(tau0)),
  interval = c(-1.5, 1.5),
  freq = NULL,
  n.cores = 1,
  cl = NULL
)
```

Arguments

`y`	vector or matrix of time series (if matrix, `nrow(y)` = length of time series)
`y.qser`	matrix or array of pre-calculated QSER at `tau0` (default = `NULL`: compute from `y` and `tau0`); if supplied, `y` can be left unspecified
`tau0`	quantile levels used to computer QSER (must be supplied with or without `y.qser`)
`tau`	quantile levels for evaluation (`min(tau0) <= tau <= max(tau0)`; default = `tau0`)

p	order of SAR model (default = NULL: automatically selected by AIC)
order.max	maximum order for AIC if p = NULL (default = NULL: determined by stats::ar())
spar	penalty parameter alla stats::smooth.spline() (default = NULL: automatically selected)
method	criterion for penalty parameter selection: "GCV" (default), "AIC", or "BIC"
type	type of quantile smoothing for residual variance: 1 = direct (default) or 2 = square root
weights	sequence of weights (length(weights) = length(tau0); default: weights = rep(1,length(tau0)))
interval	interval for spar optimization (default: interval = c(-1.5,1.5))
freq	sequence of frequencies in [0,1) (default = NULL: all Fourier frequencies)
n.cores	number of cores for parallel computing of QDFT if y.qser = NULL (default = 1)
cl	pre-existing cluster for repeated parallel computing of QDFT (default = NULL)

Value

a list with the following elements:

spec	matrix or array of SAR quantile spectrum
freq	sequence of frequencies
fit	object of SAR model
qser	matrix or array of quantile series if y.qser = NULL

Examples

```
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
tau <- seq(0.1,0.9,0.05)
n <- length(y1)
ff <- c(0:(n-1))/n
sel.f <- which(ff > 0 & ff < 0.5)
# compute from time series
y.sar <- qspec.sar(cbind(y1,y2),tau0=tau,p=1)
qfa.plot(ff[sel.f],tau,Re(y.sar$spec[1,1,sel.f,]))
# compute from quantile series
y.qser <- qser(cbind(y1,y2),tau)
y.sar <- qspec.sar(y.qser=y.qser,tau0=tau,p=1)
qfa.plot(ff[sel.f],tau,Re(y.sar$spec[1,1,sel.f,]))
```

qspec2qcoh *Quantile Coherence Spectrum*

Description

This function computes quantile coherence spectrum (QCOH) from quantile spectrum of multiple time series.

Usage

```
qspec2qcoh(qspec, k = 1, kk = 2)
```

Arguments

qspec	array of quantile spectrum
k	index of first series (default = 1)
kk	index of second series (default = 2)

Value

matrix of quantile coherence evaluated at Fourier frequencies in (0,0.5)

Examples

```
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
tau <- seq(0.1,0.9,0.05)
n <- length(y1)
ff <- c(0:(n-1))/n
sel.f <- which(ff > 0 & ff < 0.5)
y.qacf <- qacf(cbind(y1,y2),tau)
y.qper.lw <- qspec.lw(y.qacf=y.qacf,M=5)$spec
y.qcoh <- qspec2qcoh(y.qper.lw,k=1,kk=2)
qfa.plot(ff[sel.f],tau,y.qcoh)
```

sar.eq.bootstrap *Bootstrap Simulation of SAR Coefficients for Testing Equality of Granger-Causality in Two Samples*

Description

This function simulates bootstrap samples of selected spline autoregression (SAR) coefficients for testing equality of Granger-causality in two samples based on their SAR models under H0: effect in each sample equals the average effect.

Usage

```
sar.eq.bootstrap(
  y.qser,
  fit,
  fit2,
  index = c(1, 2),
  nsim = 1000,
  method = c("ar", "sar"),
  refit = FALSE,
  n.cores = 1,
  mthreads = FALSE,
  seed = 1234567
)
```

Arguments

y.qser	matrix or array of QSER from qser() or qspec.sar()$qser
fit	object of SAR model from qser2sar() or qspec.sar()$fit
fit2	object of SAR model for the other sample
index	a pair of component indices for multiple time series or a sequence of lags for single time series (default = c(1,2))
nsim	number of bootstrap samples (default = 1000)
method	method of residual calculation: "ar" (default) or "sar"
refit	if TRUE the models are refit under H0 (default=FALSE)
n.cores	number of cores for parallel computing (default = 1)
mthreads	if FALSE (default), set RhpcBLASctl::blas_set_num_threads(1)
seed	seed for random sampling (default = 1234567)

Value

array of simulated bootstrap samples of selected SAR coefficients

Examples

```
y11 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y21 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
y12 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y22 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y1.sar <- qspec.sar(cbind(y11,y21),tau0=tau,p=1)
y2.sar <- qspec.sar(cbind(y12,y22),tau0=tau,p=1)
A1.sim <- sar.eq.bootstrap(y1.sar$qser,y1.sar$fit,y2.sar$fit,index=c(1,2),nsim=5)
A2.sim <- sar.eq.bootstrap(y2.sar$qser,y2.sar$fit,y1.sar$fit,index=c(1,2),nsim=5)
```

`sar.eq.test` *Wald Test and Confidence Band for Equality of Granger-Causality in Two Samples*

Description

This function computes Wald test and confidence band for equality of Granger-causality in two samples using bootstrap samples generated by `sar.eq.bootstrap()` based on the spline autoregression (SAR) models of quantile series (QSER).

Usage

```
sar.eq.test(A1, A1.sim, A2, A2.sim, sel.lag = NULL, sel.tau = NULL)
```

Arguments

`A1`	matrix of selected SAR coefficients for sample 1
`A1.sim`	simulated bootstrap samples from `sar.eq.bootstrap()` for sample 1
`A2`	matrix of selected SAR coefficients for sample 2
`A2.sim`	simulated bootstrap samples from `sar.eq.bootstrap()` for sample 2
`sel.lag`	indices of time lags for Wald test (default = `NULL`: all lags)
`sel.tau`	indices of quantile levels for Wald test (default = `NULL`: all quantiles)

Value

a list with the following elements:

`test`	list of Wald test result containing `wald` and `p.value`
`D.u`	matrix of upper limits of 95% confidence band for `A1 - A2`
`D.l`	matrix of lower limits of 95% confidence band for `A1 - A2`

Examples

```
y11 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y21 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
y12 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y22 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y1.sar <- qspec.sar(cbind(y11,y21),tau0=tau,p=1)
y2.sar <- qspec.sar(cbind(y12,y22),tau0=tau,p=1)
A1.sim <- sar.eq.bootstrap(y1.sar$qser,y1.sar$fit,y2.sar$fit,index=c(1,2),nsim=5)
A2.sim <- sar.eq.bootstrap(y2.sar$qser,y2.sar$fit,y1.sar$fit,index=c(1,2),nsim=5)
A1 <- sar.gc.coef(y1.sar$fit,index=c(1,2))
A2 <- sar.gc.coef(y2.sar$fit,index=c(1,2))
test <- sar.eq.test(A1,A1.sim,A2,A2.sim,sel.lag=NULL,sel.tau=NULL)
```

`sar.gc.bootstrap` *Bootstrap Simulation of SAR Coefficients for Granger-Causality Analysis*

Description

This function simulates bootstrap samples of selected spline autoregression (SAR) coefficients for Granger-causality analysis based on the SAR model of quantile series (QSER) under H0: (a) for multiple time series, the second series specified in `index` is not causal for the first series specified in `index`; (b) for single time series, the series is not causal at the lags specified in `index`.

Usage

```
sar.gc.bootstrap(
  y.qser,
  fit,
  index = c(1, 2),
  nsim = 1000,
  method = c("ar", "sar"),
  refit = FALSE,
  n.cores = 1,
  mthreads = FALSE,
  seed = 1234567
)
```

Arguments

`y.qser`	matrix or array of QSER from `qser()` or `qspec.sar()$qser`
`fit`	object of SAR model from `qser2sar()` or `qspec.sar()$fit`
`index`	a pair of component indices for multiple time series or a sequence of lags for single time series (default = `c(1,2)`)
`nsim`	number of bootstrap samples (default = 1000)
`method`	method of residual calculation: `"ar"` (default) or `"sar"`
`refit`	if `TRUE` the SAR model is refit under H0 (default = `FALSE`)
`n.cores`	number of cores for parallel computing (default = 1)
`mthreads`	if `FALSE` (default), set `RhpcBLASctl::blas_set_num_threads(1)`
`seed`	seed for random sampling (default = `1234567`)

Value

array of simulated bootstrap samples of selected SAR coefficients

Examples

```
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.sar <- qspec.sar(cbind(y1,y2),tau0=tau,p=1)
A.sim <- sar.gc.bootstrap(y.sar$qser,y.sar$fit,index=c(1,2),nsim=5)
```

`sar.gc.coef` *Extraction of SAR Coefficients for Granger-Causality Analysis*

Description

This function extracts the spline autoregression (SAR) coefficients from an SAR model for Granger-causality analysis. See `sar.gc.bootstrap` for more details regarding the use of `index`.

Usage

```
sar.gc.coef(fit, index = c(1, 2))
```

Arguments

`fit`	object of SAR model from `qser2sar()` or `qspec.sar()$fit`
`index`	a pair of component indices for multiple time series or a sequence of lags for single time series (default = `c(1,2)`)

Value

matrix of selected SAR coefficients (number of lags by number of quantiles)

Examples

```
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.sar <- qspec.sar(cbind(y1,y2),tau0=tau,p=1)
A <- sar.gc.coef(y.sar$fit,index=c(1,2))
```

`sar.gc.test` *Wald Test and Confidence Band for Granger-Causality Analysis*

Description

This function computes Wald test and confidence band for Granger-causality using bootstrap samples generated by `sar.gc.bootstrap()` based the spline autoregression (SAR) model of quantile series (QSER).

Usage

```
sar.gc.test(A, A.sim, sel.lag = NULL, sel.tau = NULL)
```

Arguments

`A`	matrix of selected SAR coefficients
`A.sim`	simulated bootstrap samples from `sar.gc.bootstrap()`
`sel.lag`	indices of time lags for Wald test (default = `NULL`: all lags)
`sel.tau`	indices of quantile levels for Wald test (default = `NULL`: all quantiles)

Value

a list with the following elements:

`test`	list of Wald test result containing `wald` and `p.value`
`A.u`	matrix of upper limits of 95% confidence band of `A`
`A.l`	matrix of lower limits of 95% confidence band of `A`

Examples

```
y1 <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
y2 <- stats::arima.sim(list(order=c(1,0,0), ar=-0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.sar <- qspec.sar(cbind(y1,y2),tau0=tau,p=1)
A <- sar.gc.coef(y.sar$fit,index=c(1,2))
A.sim <- sar.gc.bootstrap(y.sar$qser,y.sar$fit,index=c(1,2),nsim=5)
y.gc <- sar.gc.test(A,A.sim)
```

sqdft *Spline Quantile Discrete Fourier Transform (SQDFT) of Time Series*

Description

This function computes spline quantile discrete Fourier transform (SQDFT) for univariate or multivariate time series through trigonometric spline quantile regression.

Usage

```
sqdft(
  y,
  tau,
  tau0 = tau,
  spar = NULL,
  w = rep(1, length(tau0)),
  criterion = c("AIC", "BIC", "GIC"),
  method = c("sqr", "sqr1", "sqr3"),
  ztol = NULL,
  solver = NULL,
  interval = NULL,
  all.knots = FALSE,
  control = list(),
  n.cores = 1,
  cl = NULL
)
```

Arguments

y	vector or matrix of time series (if matrix, nrow(y) = length of time series)
tau	sequence of quantile levels for evaluation
tau0	sequence of quantile levels for fitting (min(tau0) <= tau <= max(tau0); default = tau)
spar	smoothing parameter, selected automatically by criterion if spar = NULL or if length(spar) > 1
w	weight sequence in penalty (default = rep(1,length(tau0)))
criterion	criterion for smoothing parameter selection: "AIC" (default), "BIC", or "GIC"
method	'sqr'(default), 'sqr1', or 'sqr3'
ztol	zero-tolerance parameter to determine the model complexity (default = NULL: set internally to 1e-5 for SQR and SQR1 or 1e-4 for SQR3)
solver	'fnb' or 'sfn' for SQR and SQR1; 'piqp' or 'osqp' for SQR3 (default = NULL: set internally to 'fnb' for SQR and SQR1 or 'piqp' for SQR3)
interval	interval for spar optimization (default: c(-1.5,1.5) for SQR and SQR1 or c(0,2.5) for SQR3)

all.knots	TRUE or FALSE (default), as in stats::smooth.spline()
control	list of control parameters for QP solvers 'piqp' and 'osqp' (default = list())
n.cores	number of cores for parallel computing (default = 1)
cl	pre-existing cluster for repeated parallel computing (default = NULL)

Value

A list with the following elements:

coefficients	matrix of regression coefficients
qdft	matrix or array of the spline quantile discrete Fourier transform of y
crit	criteria for smoothing parameter selection: (AIC,BIC,GIC)
nit	maximum number of iterations
spar	optimal value of smoothing parameter

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
y.sqdft <- sqdft(y,tau,spar=0.2,method="sqr1")$qdft
plot(y.sqdft[,2])
```

---

sqdft.fit *Spline Quantile Discrete Fourier Transform (SQDFT) of Time Series Given Smoothing Parameter*

Description

This function computes spline quantile discrete Fourier transform (SQDFT) for univariate or multivariate time series through trigonometric spline quantile regression with user-supplied spar.

Usage

```
sqdft.fit(
  y,
  tau,
  tau0 = tau,
  spar = 1,
  w = rep(1, length(tau0)),
  method = c("sqr", "sqr1", "sqr3"),
  ztol = NULL,
  solver = NULL,
  all.knots = FALSE,
  control = list(),
  n.cores = 1,
  cl = NULL
)
```

Arguments

y	vector or matrix of time series (if matrix, nrow(y) = length of time series)
tau	sequence of quantile levels for evaluation
tau0	sequence of quantile levels for fitting (min(tau0) <= tau <= max(tau0); default = tau)
spar	smoothing parameter (default = 1)
w	weight sequence in penalty (default = rep(1,length(tau0)))
method	'sqr' (default), "sqr1", or 'sqr3'
ztol	zero-tolerance parameter to determine the model complexity (default = NULL: set internally to 1e-5 for SQR and SQR1 or 1e-4 for SQR3)
solver	'fnb' or 'sfn' for SQR and SQR1; 'piqp' or 'osqp' for SQR3 (default = NULL: set internally to 'fnb' for SQR and SQR1 or 'piqp' for SQR3)
all.knots	TRUE or FALSE (default), as in stats::smooth.spline()
control	list of control parameters for QP solvers 'piqp' and 'osqp' (default = list())
n.cores	number of cores for parallel computing (default = 1)
cl	pre-existing cluster for repeated parallel computing (default = NULL)

Value

A list with the following elements:

coefficients	matrix of regression coefficients
qdft	matrix or array of the spline quantile discrete Fourier transform of y
crit	criteria for smoothing parameter selection: (AIC,BIC,GIC)
nit	maximum number of iterations

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
tau0 <- seq(0.1,0.9,0.2)
y.sqdft <- sqdft.fit(y,tau,tau0=tau0,spar=0.2,method="sqr1")$qdft
```

---

sqr *Spline Quantile Regression by Formula*

Description

This function computes the spline quantile regression solution SQR, SQR1, or SAR3 on a given set of quantile levels from a regression formula with or without user-supplied smoothing parameter. SQR represents the regression coefficients as cubic spline functions of the quantile level and employs the L1-norm of their second derivative as roughness penalty. SQR1 represents the regression coefficients as linear spline functions and employs the total variation of their first derivatives as roughness penalty. SQR3 also represents the regression coefficients as cubic spline functions but employs the L2-norm of their second derivatives as roughness penalty. SQR and SQR1 are solved as linear program (LP) using `sqr.fit()` and `sqr1.fit()`, respectively. SQR3 is solved as quadratic program (QP) using `sqr3.fit()`.

Usage

```
sqr(
  formula,
  tau = seq(0.1, 0.9, 0.2),
  tau0 = tau,
  spar = NULL,
  w = rep(1, length(tau0)),
  mthreads = FALSE,
  criterion = c("AIC", "BIC", "GIC"),
  method = c("sqr", "sqr1", "sqr3"),
  type = c("dual", "primal"),
  ztol = NULL,
  solver = NULL,
  interval = NULL,
  npar = c(1, 2),
  all.knots = FALSE,
  control = list(),
  data,
  subset,
  na.action,
  model = TRUE
)
```

Arguments

`formula`	formula object, with the response on the left of a ~ operator, and the terms, separated by + operators, on the right.
`tau`	sequence of quantile levels for evaluation
`tau0`	sequence of quantile levels for fitting (`min(tau0) <= tau <= max(tau0)`; default = `tau`)
`spar`	smoothing parameter, selected automatically by `criterion` if `spar = NULL` (default)
`w`	weight sequence in penalty (default = `rep(1,length(tau0))`)
`mthreads`	if `FALSE` (default), set `RhpcBLASctl::blas_set_num_threads(1)`
`criterion`	criterion for smoothing parameter selection (`"AIC"`, `"BIC"`, or `"GIC"`)

method	'sqr' (default), 'sqr1', or 'sqr3'
type	type of QP formulation for SQR3: 'dual' (default) or 'primal'
ztol	zero-tolerance parameter to determine the model complexity (default = NULL: set internally to 1e-5 for SQR and SQR1 or 1e-4 for SQR3)
solver	'fnb' or 'sfn' for SQR and SQR1; 'piqp' or 'osqp' for SQR3 (default = NULL: set internally to 'fnb' for SQR and SQR1 or 'piqp' for SQR3)
interval	interval for spar optimization (default: c(-1.5,1.5) for SQR and SQR1 or c(1.0,2.5) for SQR3)
npar	experimental parameter (default = 1)
all.knots	TRUE or FALSE (default), same as in stats::smooth.spline()
control	list of control parameters for QP solvers 'piqp' and 'osqp'
data	data.frame object containing the observations
subset	an optional vector specifying a subset of observations to be used
na.action	a function to filter missing data (see rq() in the 'quantreg' package)
model	if TRUE then the model frame is returned (needed for calling summary subsequently)

Value

object of sqr.fit(), sqr1.fit(), or sqr3.fit()

Examples

```
library(quantreg)
data(engel)
engel$income <- (engel$income - mean(engel$income))/1000
tau <- seq(0.1,0.9,0.05)
fit <- rq(foodexp ~ income,tau=tau,data=engel)
fit.sqr1 <- sqr(foodexp ~ income,tau=tau,spar=0.5,method="sqr1",data=engel)
fit.sqr3 <- sqr(foodexp ~ income,tau=tau,spar=0.5,method="sqr3",data=engel)
par(mfrow=c(1,1),pty="m",lab=c(10,10,2),mar=c(4,4,2,1)+0.1,las=1)
plot(tau,fit$coef[2,],xlab="Quantile Level",ylab="Coeff1")
lines(tau,fit.sqr1$coef[2,])
lines(tau,fit.sqr3$coef[2,],col=2)
```

---

sqr.fit *Cubic Spline Quantile Regression with L1-Norm Roughness Penalty (SQR)*

Description

This function computes spline quantile regression solution with cubic splines and L1-norm roughness penalty (SQR) from the response vector and the design matrix on a given set of quantile levels. It uses the FORTRAN code rqfnb.f in the "quantreg" package with the kind permission of Dr. R. Koenker or the R function rq.fit.sfn() in the same package as a sparse-matrix alternative. Both solve the SQR problem as a linear program (LP).

Usage

```
sqr.fit(
  X,
  y,
  tau,
  tau0 = tau,
  spar = 1,
  w = rep(1, length(tau0)),
  mthreads = TRUE,
  ztol = 1e-05,
  solver = c("fnb", "sfn"),
  all.knots = FALSE
)
```

Arguments

X	design matrix (requirement: nrow(X) = length(y))
y	response vector
tau	sequence of quantile levels for evaluation
tau0	sequence of quantile levels for fitting (min(tau0) <= tau <= max(tau0); default = tau)
spar	smoothing parameter (default = 1)
w	weight sequence in penalty (default = rep(1,length(tau0)))
mthreads	if FALSE, set RhpcBLASctl::blas_set_num_threads(1) (default = TRUE)
ztol	zero-tolerance parameter to determine the model complexity (default = 1e-05)
solver	LP solver: 'fnb' (defaut) or 'sfn'
all.knots	TRUE or FALSE (default), same as in smooth.spline()

Value

A list with the following elements:

coefficients	matrix of regression coefficients
derivatives	matrix of derivatives of regression coefficients
crit	criteria values for spar selection: (AIC,BIC,GIC)
np	sequence of complexity measure as the number of effective parameters
fid	sequence of fidelity measure as the quasi-likelihood
info	convergence status
nit	number of iterations
K	number of spline basis functions

`sqr.fit.optim` *Cubic Spline Quantile Regression with L1-Norm Roughness Penalty (SQR) Computed by Gradient Algorithms*

Description

This function computes spline quantile regression with cubic splines and L1-norm roughness penalty by a gradient algorithm BFGS, ADAM, or GRAD.

Usage

```
sqr.fit.optim(
  X,
  y,
  tau,
  spar = 0,
  d = 1,
  weighted = FALSE,
  method = c("BFGS", "ADAM", "GRAD"),
  beta.rq = NULL,
  theta0 = NULL,
  spar0 = NULL,
  sg.rate = c(1, 1),
  mthreads = TRUE,
  control = list(trace = 0)
)
```

Arguments

`X`	vecor or matrix of explanatory variables (including intercept)
`y`	vector of dependent variable
`tau`	sequence of quantile levels in (0,1)
`spar`	smoothing parameter
`d`	subsampling rate of quantile levels (default = 1)
`weighted`	if `TRUE`, penalty function is weighted (default = `FALSE`)
`method`	optimization method: `"BFGS"` (default), `"ADAM"`, or `"GRAD"`
`beta.rq`	matrix of regression coefficients from `quantreg::rq(y~X)` for initialization (default = `NULL`)
`theta0`	initial value of spline coefficients (default = `NULL`)
`spar0`	smoothing parameter for `stats::smooth.spline()` to smooth `beta.rq` for initilaiztion (default = `NULL`)
`sg.rate`	vector of sampling rates for quantiles and observations in stochastic gradient version of GRAD and ADAM
`mthreads`	if `FALSE`, set `RhpcBLASctl::blas_set_num_threads(1)` (default = `TRUE`)

control	list of control parameters
	maxit: max number of iterations (default = 100)
	stepsize: stepsize for ADAM and GRAD (default = 0.01)
	warmup: length of warmup phase for ADAM and GRAD (default = 70)
	stepupdate: frequency of update for ADAM and GRAD (default = 20)
	stepredn: stepsize discount factor for ADAM and GRAD (default = 0.2)
	line.search.type: line search option (1,2,3,4) for GRAD (default = 1)
	line.search.max: max number of line search trials for GRAD (default = 1)
	seed: seed for stochastic version of ADAM and GRAD (default = 1000)
	trace: -1 return results from all iterations, 0 (default) return final result

Value

A list with the following elements:

beta	matrix of regression coefficients
all.beta	coefficients from all iterations for GRAD and ADAM
spars	smoothing parameters from stats::smooth.spline() for initialization
fit	object from the optimization algorithm

Examples

```
data(engel)
y <- engel$foodexp
X <- cbind(rep(1,length(y)),engel$income-mean(engel$income))
tau <- seq(0.1,0.9,0.05)
fit.rq <- quantreg::rq(y ~ X[,2],tau)
fit.sqr <- sqr(y ~ X[,2],tau,spar=0.2)
fit <- sqr.fit.optim(X,y,tau,spar=0.2,d=2,method="BFSG",beta.rq=fit.rq$coef)
fit <- sqr.fit.optim(X,y,tau,spar=0.2,d=2,method="BFSG",beta.rq=fit.rq$coef)
par(mfrow=c(1,2),pty="m",lab=c(10,10,2),mar=c(4,4,2,1)+0.1,las=1)
for(j in c(1:2)) {
  plot(tau,fit.rq$coef[j,],type="n",xlab="QUANTILE LEVEL",ylab=paste0("COEFF",j))
  points(tau,fit.rq$coef[j,],pch=1,cex=0.5)
  lines(tau,fit.sqr$coef[j,],lty=1); lines(tau,fit$beta[j,],lty=2,col=2)
}
```

---

sqr.plot *Plot of Spline Quantile Regression Coefficients*

Description

This function plots one or all regression coefficients from quantile regression (QR) and spline quantile regression (SQR) on a given sequence of quantiles against the quantile level.

Usage

```
sqr.plot(
  summary.rq,
  summary.sqr = NULL,
  summary.sqrb = NULL,
  coef.sim = NULL,
  type = "l",
  lty = c(1, 2),
  lwd = c(1.5, 1),
  cex = 0.25,
  pch = 1,
  col = c(2, 1),
  idx = NULL,
  plot.rq = TRUE,
  plot.rq.line = FALSE,
  plot.zero = FALSE,
  plot.ls = TRUE,
  plot.ci = TRUE,
  var.names = NULL,
  Ylim = NULL,
  xlim = NULL,
  set.par = TRUE,
  mfrow = NULL,
  lab = c(10, 7, 7),
  mar = c(2, 3, 2, 1) + 0.1,
  las = 1
)
```

Arguments

summary.rq	output of summary() for the QR fit from quantreg::rq()
summary.sqr	output of summary() for the primary SQR fit from sqr() (defalty = NULL)
summary.sqrb	output of summary() for the secondary SQR fit from sqr() (default = NULL)
coef.sim	output coefficients of boot.sqr() for the primary SQR (default = NULL)
type	as type parameter in plot() for plotting SQR (default = "l")
lty	line types for the primary and secondary SQR (default = c(1,2))
lwd	line widths for the primary and secondary SQR (default = c(1.5,1))
cex	as cex parameter in plot()
pch	as pch parameter in plot() for the QR (default = 1)
col	line colors for the primary and secondary SQR (default = c(2,1))
idx	index of individual coefficient to be plotted (default = NULL)
plot.rq	TRUE (default) or FALSE: if TRUE, plot QR as points
plot.rq.line	TRUE or FALSE (default): if TRUE, add line plot of QR
plot.zero	TRUE or FALSE (default): if TRUE, add zero line

plot.ls	TRUE (default) or FALSE: if TRUE, add least-square estimate with 90-percent CI
plot.ci	TRUE (default) or FALSE: if TRUE, add 90-percent bootstrap CI for the primary SQR when coef.sim is supplied or approximate CI when coef.sim = NULL
var.names	user-supplied names of regression coefficients (including the intercept)
Ylim	user-supplied matrix of ylim for each coefficient (default = NULL)
xlim	user-supplied xlim (default = NULL)
set.par	TRUE (default) or FALSE if TRUE, reset par()
mfrow	parameter for resetting par() (default = NULL)
lab	parameter for resetting par() (default = c(10,7,7))
mar	parameter for resetting par() (default = c(2,3,2,1)+0.1)
las	parameter for resetting par() (default = 1)

Value

Ylim (invisible)

`sqr1.fit` *Linear Spline Quantile Regression with Total-Variation Roughness Penalty (SQR1 or Linear SQR)*

Description

This function computes spline quantile regression with linear splines and total-variation roughness penalty (SQR1 or linear SQR) from the response vector and the design matrix on a given set of quantile levels. It uses the FORTRAN code rqfnb.f in the "quantreg" package with the kind permission of Dr. R. Koenker or the R function rq.fit.sfn() in the same package as a sparse-matrix alternative. Both solve the SQR1 problem as a linear program (LP).

Usage

```
sqr1.fit(
  X,
  y,
  tau,
  tau0 = tau,
  spar = 1,
  w = rep(1, length(tau0) - 1),
  mthreads = FALSE,
  ztol = 1e-05,
  solver = c("fnb", "sfn"),
  npar = c(1, 2),
  all.knots = FALSE
)
```

Arguments

X	design matrix (nrow(X) = length(y))
y	response vector
tau	sequence of quantile levels for evaluation
tau0	sequence of quantile levels for fitting (min(tau0) <= tau <= max(tau0); default = tau)
spar	smoothing parameter (default = 1)
w	weight sequence in penalty (default = rep(1,length(tau0)-1))
mthreads	if FALSE (default), set RhpcBLASctl::blas_set_num_threads(1)
ztol	zero-tolerance parameter to determine the model complexity (default = 1e-05)
solver	LP solver: 'fnb' (defaut) or 'sfn'
npar	experimental parameter (default = 1)
all.knots	TRUE or FALSE (default), same as in stats::smooth.spline()

Value

A list with the following elements:

coefficients	matrix of regression coefficients
derivatives	matrix of derivatives of regression coefficients
crit	sequence critera for smoothing parameter select: (AIC,BIC,GIC)
np	sequence of complexity measure as the number of effective parameters
fid	sequence of fidelity measure as the quasi-likelihood
nit	number of iterations
info	convergence status
K	number of spline basis functions

---

sqr3.fit *Cubic Spline Quantile Regression with L2-Norm Roughness Penalty (SQR3 or Cubic SQR)*

Description

This function computes spline quantile regression with cubic splines and L2-norm roughness penalty (SQR3 or cubic SQR) from the response vector and the design matrix on a given set of quantile levels. It uses solve_osqp() in the "osqp" package or solve_piqp() in the "piqp" package. Both are general-purpose quadratic program (QP) solvers in the sparse-matrix form.

Usage

```
sqr3.fit(
  X,
  y,
  tau,
  tau0 = tau,
  spar = 1,
  w = rep(1, length(tau0)),
  mthreads = FALSE,
  ztol = 1e-04,
  type = c("dual", "primal"),
  solver = c("piqp", "osqp"),
  npar = c(1, 2),
  all.knots = FALSE,
  control = list()
)
```

Arguments

X	design matrix (requirement: nrow(X) = length(y))
y	response vector
tau	sequence of quantile levels for evaluation
tau0	sequence of quantile levels for fitting (min(tau0) <= tau <= max(tau0); default = tau)
spar	smoothing parameter (default = 1)
w	weight sequence in penalty (default = rep(1,length(tau0)))
mthreads	if FALSE (default), set RhpcBLASctl::blas_set_num_threads(1)
ztol	zero-tolerance parameter to determine the model complexity (default = 1e-04)
type	type of QP formulation: 'dual' (default) or 'primal'
solver	QP solver: 'piqp' (default) or 'osqp'
npar	experimental parameter (default = 1)
all.knots	TRUE or FALSE (default) as in stats::smooth.spline()
control	list of control parameters for the QP solver (default = list())

Value

A list with the following elements:

coefficients	matrix of regression coefficients
derivatives	matrix of derivatives of regression coefficients
crit	sequence critera for smoothing parameter select: (AIC,BIC,GIC)
np	sequence of complexity measure as the number of effective parameters
fid	sequence of fidelity measure as the quasi-likelihood

info	convergence status
nit	number of iterations
K	number of spline basis functions

`sqr_deriv.plot` *Plot of Derivative of Spline Quantile Regression Coefficients*

Description

This function plots the derivative of one or all coefficients from spline quantile regression (SQR) together with the corresponding first difference from quantile regression (QR) on a given sequence of quantiles against the quantile level.

Usage

```
sqr_deriv.plot(
  fit,
  fit.sqr,
  fit.sqrb = NULL,
  deriv.sim = NULL,
  type = "l",
  typeb = "s",
  var.names = NULL,
  lty = c(1, 2),
  lwd = c(1.5, 1),
  cex = 0.25,
  pch = 1,
  col = c(2, 1),
  idx = NULL,
  Ylim = NULL,
  xlim = NULL,
  plot.zero = TRUE,
  plot.ci = TRUE,
  set.par = TRUE,
  mfrow = NULL,
  lab = c(10, 7, 7),
  mar = c(2, 3, 2, 1) + 0.1,
  las = 1
)
```

Arguments

fit	output of `quantreg::rq()` for the QR fit
fit.sqr	output of `sqr()` for the primary SQR fit
fit.sqrb	output of `sqr()` for the secondary SQR fit (default = `NULL`)
deriv.sim	output `derivatives` of `boot.sqr()` for the primary SQR (default = `NULL`)

type	as type parameter in plot() for the primary SQR (default = "l")
typeb	as type parameter in plot() for the secondary SQR (default = "s")
var.names	user-supplied names of regression coefficients (including the intercept)
lty	line types for the primary and secondary SQR (default = c(1,2))
lwd	line widths for the primary and secondary SQR (default = c(1.5,1))
cex	as cex parameter in plot()
pch	as pch parameter in plot() for QR(default = 1)
col	line colors for the primary and secondary SQR (default = c(2,1))
idx	index of individual coefficient to be plotted (default = NULL)
Ylim	user-supplied matrix of ylim for each coefficient (default = NULL)
xlim	user-supplied xlim (default = NULL)
plot.zero	TRUE or FALSE (default): if TRUE, add zero line
plot.ci	TRUE (default) or FALSE: if TRUE, add 90-percent bootstrap CI for the primary SQR when deriv.sim is supplied
set.par	TRUE (default) or FALSE if TRUE, reset par()
mfrow	parameter for resetting par() (default = NULL)
lab	parameter for resetting par() (default = c(10,7,7))
mar	parameter for resetting par() (default = c(2,3,2,1)+0.1)
las	parameter for resetting par() (default = 1)

Value

Ylim (invisible)

---

tqr.fit *Trigonometric Quantile Regression (TQR)*

Description

This function computes ordinary trigonometric quantile regression (TQR) for univariate time series at a single frequency.

Usage

```
tqr.fit(y, f0, tau, prepared = TRUE)
```

Arguments

y	vector of time series
f0	frequency in [0,1)
tau	sequence of quantile levels in (0,1)
prepared	if TRUE, intercept is removed and coef of cosine is doubled when f0 = 0.5

Value

object of `quantreg::rq()`

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
fit <- tqr.fit(y,f0=0.1,tau=tau)
plot(tau,fit$coef[1,],type='o',pch=0.75,xlab='QUANTILE LEVEL',ylab='TQR COEF')
```

---

tsqr.fit *Trigonometric Spline Quantile Regression (TSQR) of Time Series*

Description

This function computes trigonometric spline quantile regression (TSQR) for univariate time series at a single frequency using `sqr.fit()`, `sqr1.fit()`, or `sqr3.fit()`.

Usage

```
tsqr.fit(
  y,
  f0,
  tau,
  tau0 = tau,
  spar = 1,
  w = rep(1, length(tau0)),
  mthreads = FALSE,
  prepared = TRUE,
  method = c("sqr", "sqr1", "sqr3"),
  ztol = NULL,
  solver = NULL,
  all.knots = FALSE,
  control = list()
)
```

Arguments

`y`	time series
`f0`	frequency in [0,1)
`tau`	sequence of quantile levels for evaluation
`tau0`	sequence of quantile levels for fitting (`min(tau0) <= tau <= max(tau0)`; default = `tau`)
`spar`	smoothing parameter (default = 1)
`w`	weight sequence in penalty (default = `rep(1,length(tau0))`)

mthreads	if FALSE (default), set RhpcBLASctl::blas_set_num_threads(1)
prepared	if TRUE, intercept is removed and coef of cosine is doubled when f0 = 0.5
method	'sqr' (default), "sqr1", or 'sqr3'
ztol	a zero tolerance paramete to determine the model complexity (default = NULL: set internally to 1e-5 for SQR and SQR1 or 1e-4 for SQR3)
solver	'fnb' or 'sfn' for SQR and SQR1; 'piqp' or 'osqp' for SQR3 (default = NULL: set internally to 'fnb'for SQR and SQR1 or 'piqp' for SQR3)
all.knots	TRUE or FALSE (default), as in stats::smooth.spline()
control	list of control parameters for QP solvers 'piqp' and 'osqp'

Value

object of sqr.fit(), sqr1.fit(), or sqr3.fit()

Examples

```
y <- stats::arima.sim(list(order=c(1,0,0), ar=0.5), n=64)
tau <- seq(0.1,0.9,0.05)
tau0 <- seq(0.1,0.9,0.2)
fit <- tqr.fit(y,f0=0.1,tau=tau)
fit.sqr1 <- tsqr.fit(y,f0=0.1,tau=tau,tau0=tau0,spar=0.2,method='sqr1')
fit.sqr3 <- tsqr.fit(y,f0=0.1,tau=tau,tau0=tau0,spar=1,method='sqr3')
plot(tau,fit$coef[1,],type='p',xlab='QUANTILE LEVEL',ylab='TQR COEF')
lines(tau,fit.sqr1$coef[1,])
lines(tau,fit.sqr3$coef[1,],col=2)
```

---

yearssn *Yearly sunspot numbers v1.0*

Description

Yearly mean total sunspot numbers from 1700 to 2007.

Usage

```
data(yearssn)
```

Format

An object of class data.frame with 308 rows and 2 columns.

Source

Original data https://www.sidc.be/SILSO/versionarchive from WDC-SILSO, Royal Observatory of Belgium, Brussels.

References

WDC-SILSO, Royal Observatory of Belgium, Brussels. doi:10.24414/stczkc45

yearssn2 *Yearly sunspot numbers v2.0*

Description

Yearly mean total sunspot numbers from 1700 to 2024. A revised version replacing v1.0.

Usage

```
data(yearssn2024)
```

Format

An object of class `data.frame` with 325 rows and 2 columns.

Source

Original data https://www.sidc.be/SILSO/datafiles from WDC-SILSO, Royal Observatory of Belgium, Brussels.

References

WDC-SILSO, Royal Observatory of Belgium, Brussels. doi:10.24414/qnzaac80

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